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MECHANICS OF DEFORMABLE DIRECTED SURFACES

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Abstract—The paper develops a simple, yet complete and consistent non-linear dynamic thermo-elastic shell theory based on the concepts of a two-dimensional directed continuum. The constitutive equations are postulated in a general form. Stress-strain relations are derived using the equation of energy balance and the two entropy production inequalities, which are rewritten in a special form. The linear theory is discussed in detail. The structure of the stress-strain relations is presented in the most general form and then specialized to concrete examples, in particular, to the classical shell theory, using their invariance properties with respective to certain symmetry groups. The elastic moduli are determined by demanding the coincidence of the frequency dispersion surfaces obtained using the two-dimensional theory with the lowest foils of the corresponding surfaces resulting from the three-dimensional theory of elasticity.

1. INTRODUCTION

As a general rule, when speaking of the foundations of shell theory we think of the following problems: a rational derivation of the basic equations of two-dimensional shell theory from the equations of three-dimensional theory of elasticity; a formulation of the boundary conditions; an estimate of the accuracy of the equations so derived.

In so doing it is implicitly (and sometimes even explicitly) understood that the shell theory should be an asymptotically exact result of the theory of elasticity.

However, difficulties already arise in defining a shell. The intuitive definition of a shell—that of a body one dimension of which is very much smaller than the other two—is obviously inadequate. In fact, it is also necessary to restrict the class of permissible external loads[10]. Moreover, the requirement that the shell thickness be small is sometimes also superfluous. For example, the problem of pure bending of a strip is described equally well by both the three-dimensional elasticity theory and the two-dimensional shell theory, regardless of the thickness of the strip. Besides, when seen from a distance, many engineering structures look like shells and can, in fact, be considered as such, although it is impossible to define a thickness in such cases. It may also be remarked that the definition of a shell is not the main problem in the shell theory, but rather, the simplest of the remaining problems in shell theory.

For these reasons, in the past decade or so, there has probably been a revived interest in the construction of a shell theory through a direct approach based on the concept of directed deformable surfaces. The theory of such surfaces contains all the basic elements inherent in the shell theory, yet it does not coincide with the latter. Like the shell theory, the theory of directed deformable surfaces is based on the concepts of forces, couples, displacements and rotations that are specified on a two-dimensional manifold. However, in contradistinction to the shell theory, these concepts are introduced *a priori* into the theory of directed surfaces which makes it possible to avoid the contradictions inherent in the shell theory. This is undoubtedly a great merit of the theory of directed surfaces, provided, however, that it is able to cover a lot of the problems considered in the shell theory. The limits of validity of the theory of directed surfaces are as yet essentially unknown. To answer this question it will be necessary to solve the same problems as arise in constructing a shell theory. However, the construction of a theory and the determination of the range of its applicability differ significantly from each other. Whereas the question regarding the construction of the theory proper can and must be answered exactly, the same cannot be said of the latter question. The major stumbling block in the classical shell theory is probably that these two problems are mixed.

The use of a direct approach to constructing a theory of bars and shells can be traced back to Euler. Later, Dühem introduced the idea of directed spaces that was subsequently developed by

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E. and F. Cosserat. However, further development of the shell theory took a different path. It was only much later that a paper by Ericksen and Truesdell[3] dedicated to the memory of the Cosserat brothers, again directed the attention of workers towards seeking a direct approach to the construction of a shell theory. Beginning in 1965, Green and Naghdi[4] published a number of papers devoted to this subject. These authors developed a general non-isothermal theory of directed surfaces and pointed out one of the many ways of interpreting the quantities that are introduced *a priori* into such a theory, and thus illuminated a path leading from the theory of directed surfaces to a theory of shells[5]. A somewhat different approach to this problem has been presented by Reissner[11], who, however, considers it in a more restricted formulation that is very close to the linear theory of elastic shells.

The essence of the present work is very close to the works by Green and others[1, 4, 6, 8]. However, it differs from the latter, not only in purely technical details, but also as follows. Firstly, in the choice of the directors as a triad of vectors and not as a single vector. (The three vector theory has been developed in a paper by Cohen and DeSilva[2]). To this end, the present work gives a general description of the kinematics (Section 1) and the dynamics (Section 2) of a surface with such a director. Secondly, in the formulation of the second law of thermodynamics as two inequalities of Clausius–Dühem type (Section 3). This makes it possible to include the temperature drop along the shell thickness. Thus, Section 4 presents the constitutive equations and stress-strain-temperature relations for the non-isothermal theory of a directed surface.

Thirdly, in the formulation of the constitutive equations for linear coupled thermo-elasticity and in the determination of their structure. In doing so, we follow arguments similar to those of Niordson[10] and Serbin[12]. Lastly, in the determination of the elastic moduli. In this connection, it is worth pointing out that we do not use the conventional interpretation of the displacements, forces and couples in a shell as being some average of the corresponding quantities in a three-dimensional continuum. Specifically, we utilize the implicit averaging properties of frequencies of natural vibration, which, in contradistinction to displacements, rotations, forces and couples, are characteristics of the mechanical system as such and not of its deformed state.

2. KINEMATICS OF A DIRECTED SURFACE

Let us consider formally a directed surface S_t defined as follows. Let there be given a material surface, called in the sequel, a carrying surface. This can be defined by a vector function $\mathbf{R}(x^1, x^2, t) \equiv \mathbf{R}(x, t)$, where x^α† are material coordinates upon the surface and t is the time. Also, let there be given at each point of the carrying surface a triad of local vectors $\mathbf{D}_k(x, t)$, obeying the conditions

$$\mathbf{D}_k \cdot \mathbf{D}_m = \delta_{km}. \tag{2.1}$$

The vectors $\mathbf{D}_k$ are generally independent of the geometry of the carrying surface and will be called a director. The carrying surface with such a director will be called a directed surface, denoted by S_t. Thus, a directed surface at the moment t is defined if a distribution of 3-frames $\{\mathbf{R}(x, t), \mathbf{D}_k(x, t)\}$ in a 3-dimensional space is known. Such a definition of a directed surface is not intrinsic. Thus, it is advisable to define S_t in another way. To this end, let us consider two vector fields $\mathbf{R}_\alpha(x, t)$ and $\mathbf{K}_\alpha(x, t)$ with tensor components and define $\{\mathbf{R}, D_k\}$ as a solution of the set of differential equations

$$\partial_\alpha \mathbf{R} = \mathbf{R}_\alpha, \qquad \partial_\alpha \mathbf{D}_k = \mathbf{K}_\alpha \times \mathbf{D}_k,\ddagger \qquad \partial_\alpha \equiv \partial/\partial x^\alpha. \tag{2.2}$$

It is clear that $\mathbf{R}_\alpha$ and $\mathbf{K}_\alpha$ cannot be arbitrary and must obey Cartan's equation of structure[1]

$$\partial_\alpha \mathbf{R}_\beta = \partial_\beta \mathbf{R}_\alpha, \qquad \partial_\alpha \mathbf{K}_\beta - \partial_\beta \mathbf{K}_\alpha - \mathbf{K}_\alpha \times \mathbf{K}_\beta = 0. \tag{2.3}$$

These equations follow from the integrability conditions imposed on (2.2). For future use we introduce analogous conditions

$$\partial_\alpha \dot{\mathbf{R}} = (\partial_\alpha \mathbf{R})^{\boldsymbol{\cdot}}, \qquad \partial_\alpha \dot{\mathbf{D}}_k = (\partial_\alpha D_k)^{\boldsymbol{\cdot}}, \qquad \dot{f} \equiv \mathrm{d}f/\mathrm{d}t \tag{2.4}$$

†In general, in the following, latin indices take the values 1, 2, 3 and greek indices the values 1, 2.
‡The cross-product is defined in the natural basis of the carrying surface in the standard way.

for representing $\dot{\mathbf{R}}_\alpha$ and $\dot{\mathbf{K}}_\alpha$

$$\dot{\mathbf{R}}_\alpha = \partial_\alpha \mathbf{V}, \qquad \dot{\mathbf{K}}_\alpha = \partial_\alpha \boldsymbol{\Omega} + \boldsymbol{\Omega} \times \mathbf{K}_\alpha, \tag{2.5}$$

$\mathbf{V}$ being the linear velocity of the apex of 3-frame and $\boldsymbol{\Omega}$ the angular velocity of the frame about its apex, viz,

$$\dot{\mathbf{R}} = \mathbf{V}(x, t), \qquad \dot{\mathbf{D}}_k = \boldsymbol{\Omega} \times \mathbf{D}_k. \tag{2.6}$$

It can be shown that a directed surface is defined to within a rigid body motion in space if three sets of functions

$$R_{\alpha n} = \mathbf{R}_\alpha \cdot \mathbf{D}_n, \qquad K_{\alpha n} = \underset{\sim}{\mathbf{K}}_\alpha \cdot \mathbf{D}_n, \qquad B_{\alpha\beta} = -\mathbf{R}_\alpha \cdot \partial_\beta \mathbf{N} \tag{2.7}$$

($\mathbf{N}$ being a unit normal to the carrying surface) are known. Of course, these functions must satisfy the equations which follow from (2.3) and the Codazzi equation for $B_{\alpha\beta}$.

Let us introduce three tensors†

$$\underset{\sim}{\mathbf{R}} = R_{\alpha n} \mathbf{R}^\alpha \otimes \mathbf{D}^n, \qquad \underset{\sim}{\mathbf{K}} = K_{\alpha n} \mathbf{R}^\alpha \otimes \mathbf{D}^n, \qquad \underset{\sim}{\mathbf{B}} = B_{\alpha\beta} \mathbf{R}^\alpha \otimes \mathbf{R}^\beta$$
$$\mathbf{R}^\alpha \cdot \mathbf{R}_\beta = \delta_\beta{}^\alpha$$

which will be called first, second and third fundamental tensors of a directed surface, respectively. The tensors $\underset{\sim}{\mathbf{R}}$ and $\underset{\sim}{\mathbf{B}}$ are known in the classical theory of surfaces, but $\underset{\sim}{\mathbf{K}}$ is introduced by the director. As a general rule, tensor $\underset{\sim}{\mathbf{B}}$ is absent in the physical theories, as will become clear in subsequent sections. But it is present, for example, in Kirchoff's theory of shells.

Now we introduce new vectors $\boldsymbol{\Phi}_\alpha$ in place of $\mathbf{K}_\alpha$ as follows‡

$$\mathbf{K}_\alpha = \boldsymbol{\Phi}_\alpha + \underset{\sim}{\mathbf{A}} \cdot \mathbf{k}_\alpha, \qquad \underset{\sim}{\mathbf{A}}(x, t) = \mathbf{D}_k(x, t) \otimes \mathbf{d}^k(x), \tag{2.9}$$

$\underset{\sim}{\mathbf{A}}$ is an orthogonal tensor or a tensor of rotation. It can be shown that $\boldsymbol{\Phi}_\alpha(x, t)$ is defined by $\underset{\sim}{\mathbf{A}}$ and vice versa

$$\partial_\alpha \underset{\sim}{\mathbf{A}}(x, t) = \boldsymbol{\Phi}_\alpha(x, t) \times \underset{\sim}{\mathbf{A}}(x, t), \tag{2.10}$$

provided that $\underset{\sim}{\mathbf{A}}(x, 0) = 1$.

The vectors $\boldsymbol{\Phi}_\alpha$ must satisfy the equations of structure, which follow from (2.3)

$$\partial_\alpha \boldsymbol{\Phi}_\beta - \partial_\beta \boldsymbol{\Phi}_\alpha - \boldsymbol{\Phi}_\alpha \times \boldsymbol{\Phi}_\beta = 0. \tag{2.11}$$

Furthermore, we can establish that

$$\dot{\boldsymbol{\Phi}}_\alpha = \partial_\alpha \boldsymbol{\Omega} + \boldsymbol{\Omega} \times \boldsymbol{\Phi}_\alpha. \tag{2.12}$$

The vectors $\boldsymbol{\Phi}_\alpha$ preserve the advantages of $\mathbf{K}_\alpha$ because they vanish under rigid body motions of S_t. Hence, it is possible to linearize (2.11) and (2.12), which cannot, in general, be said of (2.3) and (2.5). In order to determine the orientation of the carrying surface let us introduce a tensor

$$\underset{\sim}{\mathbf{E}}(x, t) = [(\mathbf{R}_\alpha \times \mathbf{R}_\beta) \cdot \mathbf{N}] \mathbf{R}^\alpha \otimes \mathbf{R}^\beta. \tag{2.13}$$

This tensor is independent of the choice of direction of the normal $\mathbf{N}$. Indeed, if $\mathbf{N} \to -\mathbf{N}$, then $\mathbf{R}_\alpha \times \mathbf{R}_\beta \to \mathbf{R}_\alpha \times \mathbf{R}_\beta$ and $\underset{\sim}{\mathbf{E}} \to \mathbf{E}$. In concluding this section let us demonstrate a method of obtaining theory of shells. For this purpose it is sufficient to choose the director as

$$\mathbf{D}_3 = \mathbf{N}, \qquad \mathbf{D}_{(\alpha)} = \mathbf{E}_{(\alpha)}, \tag{2.14}$$

†Einstein's summation convention is adopted.

‡Here and in the sequel lower case letters stand for the corresponding functions at the moment $t = 0$. For example, $\mathbf{k}_\alpha(x) \equiv \mathbf{K}_\alpha(x, t)|_{t=0}$.

$\mathbf{E}_{(\alpha)}$ being the principal directions upon the carrying surface. For such a director there exist the relations†

$$\boldsymbol{\Omega}\times\underline{1} = (\text{Grad}\,\mathbf{V})^T - \text{Grad}\,\mathbf{V}, \qquad \text{Grad}\,\mathbf{V} \equiv \mathbf{R}^\alpha \otimes \partial_\alpha \mathbf{V}$$
$$\underline{\mathbf{K}}(x,t) = -\underline{\mathbf{B}}(x,t)\cdot\underline{\mathbf{E}}(x,t). \tag{2.15}$$

In this case we have only three degrees of freedom for every point of a directed surface.

3. STATICS AND DYNAMICS OF A DIRECTED SURFACE

It is not difficult to show that the state of "stress" in a deformed directed surface is specified by two unsymmetrical tensors $\underline{\mathbf{T}}$ and $\underline{\mathbf{M}}$ called the force tensor and the couple tensor, respectively, where

$$\underline{\mathbf{T}} = \mathbf{R}_\alpha \otimes \mathbf{T}^\alpha = T^{\alpha n}\mathbf{R}_\alpha \otimes \mathbf{D}_n, \qquad \underline{\mathbf{M}} = \mathbf{R}_\alpha \otimes \mathbf{M}^\alpha = M^{\alpha n}\mathbf{R}_\alpha \otimes \mathbf{D}_n \tag{3.1}$$

with

$$\mathbf{T}^\alpha = \sqrt{(R^{\alpha\alpha})}\mathbf{T}_{(\alpha)}, \qquad \mathbf{M}^\alpha = \sqrt{(R^{\alpha\alpha})}\mathbf{M}_{(\alpha)}, \qquad R^{\alpha\alpha} \equiv \mathbf{R}^\alpha\cdot\mathbf{R}^\alpha. \tag{3.2}$$

The vectors $\mathbf{T}_{(\alpha)}$ and $\mathbf{M}_{(\alpha)}$ are, respectively, the physical vectors of the force and of the couple acting on the coordinate curves $x^\alpha = \text{const}$.

It is readily seen that Cauchy's theorem remains valid

$$\mathbf{T}_{(\nu)} = \boldsymbol{\nu}\cdot\underline{\mathbf{T}}, \qquad \mathbf{M}_{(\nu)} = \boldsymbol{\nu}\cdot\underline{\mathbf{M}}, \tag{3.3}$$

where $\boldsymbol{\nu}$ is a unit normal vector directed outwards from a curve C on the carrying surface and satisfying the condition $\boldsymbol{\nu}\cdot\mathbf{N} = 0$. Moreover, the tensors $\underline{\mathbf{T}}$ and $\underline{\mathbf{M}}$ must satisfy the equations of motion

$$\text{Div}\,\underline{\mathbf{T}} + \rho\mathbf{F} = \rho\dot{\mathbf{V}}, \qquad \text{Div}\,\underline{\mathbf{M}} + \mathbf{R}_\alpha\times\mathbf{T}^\alpha + \rho\mathbf{L} = \rho\underline{\boldsymbol{\theta}}\cdot\dot{\boldsymbol{\Omega}} \tag{4.3}$$

where $\text{Div}\,\underline{\mathbf{S}} \equiv \mathbf{R}^\alpha\cdot\partial_\alpha\underline{\mathbf{S}}$.

The quantities $\rho\mathbf{F}$, $\rho\mathbf{L}$, ρ, $\rho\underline{\boldsymbol{\theta}}$ are called the surface force, the surface couple, the surface mass density and the surface tensor of rotary inertia ($\rho\underline{\boldsymbol{\theta}} = \rho\underline{\boldsymbol{\theta}}^T$), respectively.

4. EQUATION OF ENERGY BALANCE AND ENTROPY PRODUCTION INEQUALITIES

Let us postulate two laws of thermodynamics for the directed surface S_t. The first law or the equation of energy balance can be written as

$$\frac{d}{dt}\int_{\Delta S_t}\rho\left[\frac{1}{2}\mathbf{V}\cdot\mathbf{V} + \frac{1}{2}\boldsymbol{\Omega}\cdot\underline{\boldsymbol{\theta}}\cdot\boldsymbol{\Omega} + U\right]d\Sigma = \int_{\Delta S_t}\rho[q + \mathbf{F}\cdot\mathbf{V} + \mathbf{L}\cdot\boldsymbol{\Omega}]\,d\Sigma$$
$$+\int_C[\mathbf{T}_{(\nu)}\cdot\mathbf{V} + \mathbf{M}_{(\nu)}\cdot\boldsymbol{\Omega} - h_{(\nu)}]\,dC, \tag{4.1}$$

where U is the internal energy density, ρq- the heat supply density and $h_{(\nu)}$- the heat influx across C.

It is readily seen that (4.1) is an invariant with respect to the group of rigid body motions if the tensors $\underline{\mathbf{T}}$ and $\underline{\mathbf{M}}$ satisfy the eqns (3.4) and the following equation of mass conservation holds good

$$\frac{d}{dt}\int_{\Delta S_t}\rho\,d\Sigma = 0 \to \rho\sqrt{R} = \rho_0\sqrt{r}, \qquad \rho_0(x) = \rho(x,t)|_{t=0}. \tag{4.2}$$

†Superscript T indicates transpose.

For future use it is necessary to write down (4.1) in the form of a local energy equation. Making use of the divergence theorem

$$\int_c \boldsymbol{\nu} \cdot \underset{\sim}{\mathbf{S}}\, dc = \int_{\Delta S_t} (\mathrm{Div}\, \underset{\sim}{\mathbf{S}} + 2H\mathbf{N} \cdot \underset{\sim}{\mathbf{S}})\, d\Sigma, \tag{4.3}$$

($\underset{\sim}{\mathbf{S}}$ being an arbitrary tensor field upon the surface, H- the mean curvature of the carrying surface), the last integral in (4.1) can be rewritten as

$$\int_c [\boldsymbol{\nu} \cdot \underset{\sim}{\mathbf{T}} \cdot \mathbf{V} + \boldsymbol{\nu} \cdot \underset{\sim}{\mathbf{M}} \cdot \boldsymbol{\Omega} - \boldsymbol{\nu} \cdot \mathbf{h}]\, dC = \int_{\Delta S_t} [(\mathrm{Div}\, \underset{\sim}{\mathbf{T}}) \cdot \mathbf{V} + (\mathrm{Div}\, \underset{\sim}{\mathbf{M}}) \cdot \boldsymbol{\Omega} + \rho q$$
$$+ \underset{\sim}{\mathbf{T}}^T : \mathrm{Grad}\, \mathbf{V} + \underset{\sim}{\mathbf{M}}^T : \mathrm{Grad}\, \boldsymbol{\Omega}\text{-Div}\, \mathbf{h} - 2H\mathbf{N} \cdot \mathbf{h}]\, d\Sigma, \tag{4.4}$$

whence, by use of (3.4), (4.1) takes the necessary local form

$$\rho \dot{U} = \underset{\sim}{\mathbf{T}}^T : \mathrm{Grad}\, \mathbf{V} - (\mathbf{R}_\alpha \times \mathbf{T}^\alpha) \cdot \boldsymbol{\Omega} + \underset{\sim}{\mathbf{M}}^T : \mathrm{Grad}\, \boldsymbol{\Omega}\text{-Div}\, \mathbf{h} - 2H\mathbf{N} \cdot \mathbf{h} + \rho q. \tag{4.5}$$

However, the equation is inconvenient to use because it is expressed through terms which are not intrinsic for a directed surface. It is, therefore, necessary to further transform (4.5). In this connection, we make use of (2.6) and obtain

$$\underset{\sim}{\mathbf{T}}^T : \mathrm{Grad}\, \mathbf{V} - (\mathbf{R}_\alpha \times \mathbf{T}^\alpha) \cdot \boldsymbol{\Omega} = \underset{\sim}{\mathbf{T}}_*^T : \dot{\underset{\sim}{\mathbf{R}}}^x = \underset{\sim}{\mathbf{T}}_*^T : \dot{\underset{\sim}{\boldsymbol{\epsilon}}},$$

$\underset{\sim}{\mathbf{T}}_* = T^{\alpha n} \mathbf{r}_\alpha \otimes \mathbf{d}_n$ being an energetical force tensor, $\underset{\sim}{\mathbf{R}}^x = R_{\alpha n} \mathbf{r}^\alpha \otimes \mathbf{d}^n$-the first deformation measure of the directed surface and $\underset{\sim}{\boldsymbol{\epsilon}} = \underset{\sim}{\mathbf{R}}^T - \underset{\sim}{\mathbf{r}}$-the first deformation tensor.

In an analogous manner, by making use of (2.5), it is readily shown that

$$\underset{\sim}{\mathbf{M}}^T : \mathrm{Grad}\, \boldsymbol{\Omega} = \underset{\sim}{\mathbf{M}}_*^T : \dot{\underset{\sim}{\mathbf{K}}}^x = \underset{\sim}{\mathbf{M}}_*^T : \dot{\underset{\sim}{\boldsymbol{\Phi}}} = M^{\alpha n} \dot{\Phi}_{\alpha n},$$

$\underset{\sim}{\mathbf{M}}_* \equiv M^{\alpha n} \mathbf{r}_\alpha \otimes \mathbf{d}_n$ being an energetical couple tensor, $\underset{\sim}{\mathbf{K}}^x \equiv K_{\alpha n} \mathbf{r}^\alpha \otimes \mathbf{d}^n$-the second deformation measure of the directed surface and $\underset{\sim}{\boldsymbol{\Phi}} \equiv \underset{\sim}{\mathbf{K}}^x - \underset{\sim}{\mathbf{k}}$-the second deformation tensor ($\underset{\sim}{\boldsymbol{\Phi}} = \Phi_{\alpha n} \mathbf{r}^\alpha \otimes \mathbf{d}^n$, $\dot{\Phi}_{\alpha n} = \dot{\boldsymbol{\Phi}}_\alpha \cdot \mathbf{D}_n$).

Now the eqn (4.5) takes the form

$$\rho \dot{U} = \underset{\sim}{\mathbf{T}}_*^T : \dot{\underset{\sim}{\boldsymbol{\epsilon}}} + \underset{\sim}{\mathbf{M}}^T : \dot{\underset{\sim}{\boldsymbol{\Phi}}} + \rho q - \mathrm{Div}\, \mathbf{h} - 2H\mathbf{N} \cdot \mathbf{h}, \tag{4.6}$$

which must be satisfied by an arbitrary process.

It is useful to rewrite (4.6) as a set of two equations by introducing an additional quantity Q-the heat exchanged.

Let the directed surface S_t be situated in a heat containing medium. We shall distinguish sides 1 and 2 of S_t according to the orientation of the normal $\mathbf{N}$—the normal being directed from side 2 towards side 1. Furthermore, let us consider a point removed from the boundary region of S_t, but belonging to S_t itself and let $T_+(x, t)$ and $T_-(x, t)$ be, respectively, the temperature of the surrounding medium from the upper and lower faces of S_t in the vicinity of this point.† Then the heat supply density ρq can be represented as a sum of three terms $\rho q = \rho q_1 + \rho q_2 + \rho q_0$, where ρq_0 is the heat production "inside" S_t, ρq_1-the heat input to side 1 from the medium with temperature $T_+(x, t)$ and ρq_2-the heat input to side 2 from the medium with temperature $T_-(x, t)$, and (4.6) can be rewritten as‡

$$\rho \dot{U}_A - \underset{\sim}{\mathbf{T}}_{*A}^T : \dot{\underset{\sim}{\boldsymbol{\epsilon}}} - \underset{\sim}{\mathbf{M}}_{*A}^T : \dot{\underset{\sim}{\boldsymbol{\Phi}}} = 0.5 \rho q_0 + \rho q_A + \rho Q_A\text{-Div}\, \mathbf{h}_A - 2H\mathbf{N} \cdot \mathbf{h}_A, \tag{4.7}$$

where subscript A is assigned to the side A of S_t.

The quantity $Q \equiv Q_1 \equiv -Q_2$ will be called the exchanged heat, i.e. the heat exchanged between sides 1 and 2.

†Some work allowing for a variation of temperature through the surface has been reported by Naghdi[9].
‡The latin indices $A, B, \ldots$ take the values 1,2, no summation being intended.

Let us postulate the second law of thermodynamics in the form of two inequalities of the Clausius–Dühem type.

$$\frac{d}{dt}\int_{\Delta S_t} \rho S_1 \, d\Sigma - \int_{\Delta S_t} \rho\left[\frac{q_0}{2T_1}+\frac{q_1}{T_+}+\frac{Q_1}{T_2}\right] d\Sigma + \int_{t} \frac{h_1}{T_1} dc \geq 0, \tag{4.8}$$

ρS_A being the surface entropy density of side 1. The second inequality can be obtained by replacing subscripts 1 and 2 and exchanging T_- for T_+. In what follows, consideration is given to (4.8), the other inequality being similar. The total specific entropy of S_t can be found as $S = S_1 + S_2$. Making use of the divergence theorem, the inequality (4.8) can be written in the local form

$$\rho \dot{S}_1 - \rho q_1 \frac{T_1 - T_+}{T_1 T_+} - \rho Q_1 \frac{T_1 - T_2}{T_1 T_2} - \frac{1}{T_1}[0.5\rho q_0 + \rho q_1 + \rho Q_1 - \text{Div}\, \mathbf{h}_1 - 2H\mathbf{N}\cdot\mathbf{h}_1] - T_1^{-2}\mathbf{h}_1 \cdot \text{Grad}\, T_1 \geq 0,$$

which, by use of (4.7), takes the form

$$\rho \dot{S}_1 T_1 - \rho q_1 T_+^{-1}(T_1 - T_+) - \rho Q_1 T_2^{-1}(T_1 - T_2) - \rho \dot{U}_1 + \underset{\sim}{T}_{*1}^T : \dot{\underset{\sim}{\epsilon}} + \underset{\sim}{M}_{*1}^T : \dot{\underset{\sim}{\Phi}} - T_1^{-1}\mathbf{h}_1 \cdot \text{Grad}\, T_1 \geq 0. \tag{4.9}$$

Furthermore, by introducing Helmholtz free energy per unit mass

$$A_B = U_B - S_B T_B \tag{4.10}$$

inequality (4.9) can be written as

$$-\rho(\dot{A}_1 + S_1 \dot{T}_1) + \underset{\sim}{T}_{*1}^T : \dot{\underset{\sim}{\epsilon}} + \underset{\sim}{M}_{*1}^T : \dot{\underset{\sim}{\Phi}} - \rho q_1 T_+^{-1}(T_1 - T_+) - \rho Q_1 T_2^{-1}(T_1 - T_2) - T_1^{-1}\mathbf{h}_1 \cdot \text{Grad}\, T_1 \geq 0. \tag{4.11}$$

Equations (4.7) and the inequality (4.9) or (4.11) must be satisfied by an arbitrary process. These will form the foundations of the following.

5. NON-LINEAR THERMO-ELASTICITY OF A DIRECTED SURFACE. CONSTITUTIVE EQUATIONS AND EQUATIONS OF HEAT TRANSFER

As a general rule, the equations of energy balance and the entropy production inequalities play the role of compulsory constraints. However, in some specific cases they allow us to obtain more interesting results. Non-linear thermo-elasticity is one such particular case and is considered below. Let us treat the functions $\underset{\sim}{\epsilon}$, $\underset{\sim}{\Phi}$, T_B, Grad T_B, as the state variables. Let there be given constitutive equations for A_B, $\mathbf{h}_B$, Q_B, g_B, g_0, which are functions of the state variables.

The inequality (4.11) now take the form

$$\left(-\rho\frac{\partial A_1}{\partial \underset{\sim}{\epsilon}} + \underset{\sim}{T}_1\right)^T : \dot{\underset{\sim}{\epsilon}} + \left(-\rho\frac{\partial A_1}{\partial \underset{\sim}{\Phi}} + \underset{\sim}{M}_{*1}\right)^T : \dot{\underset{\sim}{\Phi}} - \left(\rho\frac{\partial A_1}{\partial T_1} + S_1\right)\dot{T}_1 - \rho\frac{\partial A_1}{\partial T_2}\dot{T}_2$$
$$-\rho\frac{\partial A_1}{\partial\, \text{Grad}\, T_1}\cdot\dot{\overline{\text{Grad}\, T_1}} - \rho\frac{\partial A_1}{\partial\, \text{Grad}\, T_2}\cdot\dot{\overline{\text{Grad}\, T_2}}$$
$$-\rho q_1\frac{T_1 - T_+}{T_+} - \rho Q_1\frac{T_1 - T_2}{T_2} - \frac{1}{T_1}\mathbf{h}_1\cdot\text{Grad}\, T_1 \geq 0. \tag{5.1}$$

It is clear that (5.1) is satisfied if, and only if,

$$\underset{\sim}{T}_{*B} = \rho\frac{\partial A_B}{\partial \underset{\sim}{\epsilon}};\quad \underset{\sim}{M}_{*B} = \rho\frac{\partial A_B}{\partial \underset{\sim}{\Phi}};\quad S_B = -\frac{\partial A_B}{\partial T_B};\quad \frac{\partial A_B}{\partial T_c} = 0,\quad (B \neq C); \tag{5.2}$$

$$\frac{\partial A_B}{\partial\, \text{Grad}\, T_c} = \mathbf{0};\quad -q_1(T_1 - T_+) \geq 0;\quad -Q(T_1 - T_2) \geq 0;\quad -q_2(T_2 - T_-) \geq 0;\quad -\mathbf{h}_A\cdot\text{Grad}\, T_A \geq 0. \tag{5.3}$$

Hence, the Helmholtz free energy A_1 is independent of the variables T_2, Grad T_A and A_2—that of the variables T_1, Grad T_A.

The relations (5.2) can be rewritten as

$$\underset{\sim}{\mathbf{T}}_* = \rho\frac{\partial A}{\partial \underset{\sim}{\epsilon}};\quad \underset{\sim}{\mathbf{M}}_* = \rho\frac{\partial A}{\partial \underset{\sim}{\Phi}};\quad S_B = -\frac{\partial A}{\partial T_B} \tag{5.4}$$

or

$$\underset{\sim}{\mathbf{T}} = \rho \underset{\sim}{\mathbf{C}}^T \cdot \frac{\partial A}{\partial \underset{\sim}{\epsilon}} \cdot \underset{\sim}{\mathbf{A}}^T;\quad \underset{\sim}{\mathbf{M}} = \rho \mathbf{G}^T \cdot \frac{\partial A}{\partial \underset{\sim}{\Phi}} \cdot \underset{\sim}{\mathbf{A}}^T;\quad S_B = -\frac{\partial A}{\partial T_B}, \tag{5.5}$$

where $\underset{\sim}{\mathbf{C}} = \text{grad}\,\mathbf{R} + \mathbf{n}\otimes\mathbf{N}$, grad $= \text{Grad}|_{t=0}$, and $\underset{\sim}{\mathbf{A}}$ is specified by (2.9). Of course, we could treat the functions $\underset{\sim}{\epsilon}$, $\underset{\sim}{\Phi}$, S_1, S_2 as the state variables. In this case the inequalities (5.3) would remain unchanged, but instead of (5.5) we would have

$$\underset{\sim}{\mathbf{T}} = \rho \underset{\sim}{\mathbf{C}}^T \cdot \frac{\partial U}{\partial \underset{\sim}{\epsilon}} \cdot \underset{\sim}{\mathbf{A}}^T;\quad \underset{\sim}{\mathbf{M}} = \rho \underset{\sim}{\mathbf{C}}^T \cdot \frac{\partial U}{\partial \underset{\sim}{\Phi}} \cdot \underset{\sim}{\mathbf{A}}^T;\quad T_B = \frac{\partial U}{\partial S_B}. \tag{5.6}$$

Then, by making use of (5.2), (4.10) and (4.7), we obtain the equations of heat transfer

$$\text{Div}\,\mathbf{h}_A - 2H\mathbf{N}\cdot\mathbf{h}_A = 0.5\rho g_0 + \rho(q_A + Q_A) - \rho T_A \dot{S}_A. \tag{5.7}$$

6. LINEARIZATION OF BASIC EQUATIONS

The linear theory we shall discuss is such that $\mathbf{R}_\alpha(x,t) - \mathbf{r}_\alpha(x,t)$ and $\mathbf{\Phi}_\alpha(x,t)$, as well as their spatial and time derivatives, are all infinitesimally small. In this case there is no distinction between the true and energetical force tensors. The same holds good for couple tensor. The asterisk mark "*" is thus omitted in the sequel.

The equations of motion, expressed through the undeformed metric tensor, take the form

$$\nabla\cdot\underset{\sim}{\mathbf{T}} + \rho\mathbf{F} = \rho\dot{\mathbf{V}},\quad \nabla\cdot\underset{\sim}{\mathbf{M}} + \mathbf{r}_\alpha\times T^\alpha + \rho\mathbf{L} = \rho\underset{\sim}{\boldsymbol{\theta}}\cdot\dot{\mathbf{\Omega}},\quad \nabla = \text{grad} \equiv \text{Grad}|_{t=0}. \tag{6.1}$$

The first of the eqns (2.3) can be written as

$$\partial_\alpha(\mathbf{R}_\beta - \mathbf{r}_\beta) = \partial_\beta(\mathbf{R}_\alpha - \mathbf{r}_\alpha).$$

Hence there exists a vector $\mathbf{u}$, called the displacement vector, such that

$$\mathbf{R}_\alpha - \mathbf{r}_\alpha = \partial_\alpha\mathbf{u}. \tag{6.2}$$

Moreover, if $\mathbf{\Phi}_\alpha$ are also infinitesimally small, (2.11) reduce to

$$\partial_\alpha\mathbf{\Phi}_\beta = \partial_\beta\mathbf{\Phi}_\alpha,$$

whence it follows that there exists a vector $\boldsymbol{\phi}(x,t)$-called the vector of infinitesimal rotation, such that

$$\mathbf{\Phi}_\alpha = \partial_\alpha\boldsymbol{\phi}(x,t). \tag{6.3}$$

Whereas the displacement vector $\mathbf{u}(x,t)$ exists both in the linear and in the non-linear theories, the vector $\boldsymbol{\phi}(x,t)$ has a meaning only in the former theory. Making use of (6.2), (6.3), (2.5) and (2.12), after corresponding reduction, we get

$$\mathbf{V}(x,t) = \dot{\mathbf{u}}(x,t),\quad \mathbf{\Omega}(x,t) = \dot{\boldsymbol{\phi}}(x,t). \tag{6.4}$$

It is not difficult to show that

$$\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{1}} + \boldsymbol{\phi}\times\underset{\sim}{\mathbf{1}};\quad \mathbf{D}_k = \mathbf{d}_k + \boldsymbol{\phi}\times\mathbf{d}_k. \tag{6.5}$$

Furthermore, the second and the first deformation tensors take the form

$$\underset{\sim}{\Phi} \simeq \nabla \otimes \boldsymbol{\phi} \equiv \underset{\sim}{\boldsymbol{\kappa}} \qquad \underset{\sim}{\boldsymbol{\epsilon}} \simeq \nabla \otimes \mathbf{u} + \underset{\sim}{\mathbf{r}} \times \boldsymbol{\phi} \equiv \underset{\sim}{\mathbf{e}}, \tag{6.6}$$

$\underset{\sim}{\boldsymbol{\kappa}}$, $\underset{\sim}{\mathbf{e}}$ being the infinitesimal second and first tensors, respectively.

For the sake of conciseness, the tensors $\underset{\sim}{\mathbf{e}}$ and $\underset{\sim}{\boldsymbol{\kappa}}$ will be called the strain tensor and the "bending" tensor, respectively, but we emphasize that the "bending" tensor $\underset{\sim}{\boldsymbol{\kappa}}$ can exist even though there is no bending in the real sense of the word.

In order to determine the vectors $\mathbf{u}$ and $\boldsymbol{\phi}$ from eqns (6.6), the tensors $\underset{\sim}{\mathbf{e}}$ and $\underset{\sim}{\boldsymbol{\kappa}}$ must satisfy the integrability conditions†

$$\nabla \cdot (\underset{\sim}{\boldsymbol{\pi}} \cdot \underset{\sim}{\boldsymbol{\kappa}}) = 0, \quad \nabla \cdot (\underset{\sim}{\boldsymbol{\pi}} \cdot \underset{\sim}{\mathbf{e}}) + (\underset{\sim}{\boldsymbol{\pi}} \cdot \underset{\sim}{\boldsymbol{\kappa}})_x = 0, \tag{6.7}$$

where $(\underset{\sim}{\boldsymbol{\pi}} \cdot \underset{\sim}{\boldsymbol{\kappa}})_x$ is the vector invariant of the tensor $\underset{\sim}{\boldsymbol{\pi}} \cdot \underset{\sim}{\boldsymbol{\kappa}}$, for example, $\underset{\sim}{\mathbf{T}}_x \equiv (\mathbf{r}_\alpha \otimes \mathbf{T}^\alpha)_x \equiv \mathbf{r}_\alpha \times \mathbf{T}^\alpha$.

We point out here the so-called statico-geometrical analogy, namely that there exists a duality between the homogeneous statical eqns (6.1) (when $\mathbf{F} = \mathbf{L} = \mathbf{v} = \boldsymbol{\Omega} = \mathbf{0}$) and the equations of structure (6.7). The duality can be established by the following replacement: $\underset{\sim}{\mathbf{T}} \rightleftarrows \underset{\sim}{\boldsymbol{\pi}} \cdot \underset{\sim}{\boldsymbol{\kappa}}$ and $\underset{\sim}{\mathbf{M}} \rightleftarrows \underset{\sim}{\boldsymbol{\pi}} \cdot \underset{\sim}{\mathbf{e}}$. It is thus possible to take into consideration Lur'e–Gol'denveizer stress-functions $\mathbf{t}$ and $\mathbf{m}$[6, 8]

$$\underset{\sim}{\mathbf{T}} = \underset{\sim}{\boldsymbol{\pi}} \cdot \nabla \otimes \mathbf{t}, \qquad \underset{\sim}{\mathbf{M}} = \underset{\sim}{\boldsymbol{\pi}} \cdot (\nabla \otimes \mathbf{m} + \underset{\sim}{\mathbf{r}} \times \mathbf{t}). \tag{6.8}$$

It is readily seen that the homogeneous eqns (6.1) are satisfied by arbitrary functions $\mathbf{t}$ and $\mathbf{m}$.

In order to define the functions $\mathbf{t}$ and $\mathbf{m}$ it is necessary to employ the equations of structure (6.7) and the constitutive equations. This will be discussed in the following sections.

7. STRUCTURE OF THE FREE ENERGY FOR THE LINEAR THEORY

For an infinitesimal deformation of the directed surface that is initially unstressed the free energy is a quadratic function of state variables: $\underset{\sim}{\mathbf{e}}$, $\underset{\sim}{\boldsymbol{\kappa}}$, $T_1 - T_0$, $T_2 - T_0$, where T_0 is the temperature in the natural state:

$$\begin{aligned} A &= \frac{1}{2}\underset{\sim}{\mathbf{e}} : \underset{\sim}{\mathbf{A}} : \underset{\sim}{\mathbf{e}} + \underset{\sim}{\mathbf{e}} : \underset{\sim}{\mathbf{B}} : \underset{\sim}{\boldsymbol{\kappa}} + \frac{1}{2}\underset{\sim}{\boldsymbol{\kappa}} : \underset{\sim}{\mathbf{C}} : \underset{\sim}{\boldsymbol{\kappa}} + \underset{\sim}{\mathbf{A}}_1 : \underset{\sim}{\mathbf{e}}(T_1 - T_0) + \underset{\sim}{\mathbf{A}}_2 : \underset{\sim}{\mathbf{e}}(T_2 - T_0) \\ &\quad + \underset{\sim}{\mathbf{B}}_1 : \underset{\sim}{\boldsymbol{\kappa}}(T_1 - T_0) + \underset{\sim}{\mathbf{B}}_2 : \underset{\sim}{\boldsymbol{\kappa}}(T_2 - T_0) + \frac{1}{2}K_1(T_1 - T_0)^2 + \frac{1}{2}K_2(T_2 - T_0)^2. \end{aligned} \tag{7.1}$$

The term $(T_1 - T_0)(T_2 - T_0)$ does not enter the expression because of (5.2). The fourth rank tensors $\underset{\sim}{\mathbf{A}}$, $\underset{\sim}{\mathbf{B}}$, $\underset{\sim}{\mathbf{C}}$ are defined by

$$(\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{B}}, \underset{\sim}{\mathbf{C}}) = (A, B, C)^{n\alpha m\beta}\mathbf{d}_n \otimes \mathbf{r}_\alpha \otimes \mathbf{d}_m \otimes \mathbf{r}_\beta, \quad (A, C)^{n\alpha m\beta} = (A, C)^{m\beta n\alpha} \tag{7.2}$$

and the second rank tensors $\underset{\sim}{\mathbf{A}}_1$, $\underset{\sim}{\mathbf{A}}_2$, $\underset{\sim}{\mathbf{B}}_1$, $\underset{\sim}{\mathbf{B}}_2$ by

$$(\underset{\sim}{\mathbf{A}}_1, \underset{\sim}{\mathbf{A}}_2, \underset{\sim}{\mathbf{B}}_1, \underset{\sim}{\mathbf{B}}_2) = (\mathrm{A}_{(1)}, \mathrm{A}_{(2)}, \mathrm{B}_{(1)}, \mathrm{B}_{(2)})^{n\alpha}\mathbf{d}_n \otimes r_\alpha. \tag{7.3}$$

All these tensors depend on the geometry of the directed surface and the physical properties of its material. Thus, the free energy will depend on 104 scalar functions. For certain materials the number of scalar functions may be reduced, but it still remains rather large. This follows from the fact that the local group of symmetry‡ for an arbitrary directed surface contains, at most, only three elements, which are a solution of the set of equation

$$\underset{\sim}{\mathbf{Q}} \cdot \underset{\sim}{\mathbf{k}} \cdot \underset{\sim}{\mathbf{k}}^T \cdot \underset{\sim}{\mathbf{Q}}^T = \underset{\sim}{\mathbf{k}} \cdot \underset{\sim}{\mathbf{k}}^T, \qquad \underset{\sim}{\mathbf{Q}} \cdot \underset{\sim}{\mathbf{b}}^2 \cdot \underset{\sim}{\mathbf{Q}}^T = \underset{\sim}{\mathbf{b}}^2, \qquad \underset{\sim}{\mathbf{Q}} \cdot \boldsymbol{\theta} \cdot Q^T = \boldsymbol{\theta}, \tag{7.4}$$

$\underset{\sim}{\mathbf{Q}}$ being an orthogonal tensor.

†The discrimant tensor $\underset{\sim}{\boldsymbol{\pi}}$ is defined as $\underset{\sim}{\mathbf{E}}|_{t=0} = \underset{\sim}{\boldsymbol{\pi}}$.

‡In the sequel the local group of symmetry is abbreviated as LGS.

The set of orthogonal tensors satisfying (7.4) is called the LGS for a directed surface. It is readily seen, that the LGS for a carrying surface, being the solution of $\underset{\sim}{\mathbf{Q}} \cdot \underset{\sim}{\mathbf{b}}^2 \cdot \underset{\sim}{\mathbf{Q}} = \underset{\sim}{\mathbf{b}}^2$, includes the LGS for a directed surface, but contains only three elements that are defined by

$$\underset{\sim}{\mathbf{1}};\ \underset{\sim}{\mathbf{Q}}_{(1)} = -\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{n} \otimes \mathbf{n}; \qquad \underset{\sim}{\mathbf{Q}}_{(2)} = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{n} \otimes \mathbf{n}, \tag{7.5}$$

where $\mathbf{e}_\alpha$ are the principal directions upon the carrying surface at the moment $t = 0$ and $\mathbf{n}$ is a unit normal to the carrying surface. It should be mentioned, however, that the direction of $\mathbf{n}$ is not defined—a fact that will be used later.

If we confine our attention to the specific case in which the director is initially coincident with the natural basis of the carrying surface, viz.

$$\mathbf{d}_{(\alpha)} = \mathbf{e}_{(\alpha)}, \qquad \mathbf{d}_3 = \mathbf{n}, \tag{7.6}$$

then formulae (2.5) are valid, and we have

$$\underset{\sim}{\mathbf{k}} = -\underset{\sim}{\mathbf{b}} \cdot \underset{\sim}{\boldsymbol{\pi}}, \qquad \underset{\sim}{\mathbf{k}} \cdot \underset{\sim}{\mathbf{k}}^T = \underset{\sim}{\mathbf{b}}^2. \tag{7.7}$$

Hence, instead of (7.4), we get

$$\underset{\sim}{\mathbf{Q}} \cdot \underset{\sim}{\mathbf{b}}^2 \cdot \underset{\sim}{\mathbf{Q}}^T = \underset{\sim}{\mathbf{b}}^2, \qquad \underset{\sim}{\mathbf{Q}} \cdot \underset{\sim}{\boldsymbol{\theta}} \cdot \underset{\sim}{\mathbf{Q}}^T = \underset{\sim}{\boldsymbol{\theta}}. \tag{7.8}$$

The equations involving $\underset{\sim}{\boldsymbol{\theta}}$ show that the LGS for a carrying surface will be the same as the LGS for a directed surface if, and only if, the tensor $\underset{\sim}{\boldsymbol{\theta}}$ is of diagonal form in the natural basis of the carrying surface. Furthermore, it is clear that the full LGS for a **physical** directed surface is contained in the LGS for a directed surface, but the converse is not true, because the latter group takes no regard of the elastic properties of the directed surface.

Let the physical properties of the directed surface remain unchanged during the transformations (7.5). Then it can be proved that elasticity tensors from (7.1) take the form

$$\begin{aligned}\underset{\sim}{\mathbf{A}} = {} & A_1 \underset{\sim}{\mathbf{r}} \otimes \underset{\sim}{\mathbf{r}} + A_2 \underset{\sim}{\boldsymbol{\pi}} \otimes \underset{\sim}{\boldsymbol{\pi}} + A_3 \underset{\sim}{\boldsymbol{\pi}}_1 \otimes \underset{\sim}{\boldsymbol{\pi}}_1 + A_4 \underset{\sim}{\mathbf{h}} \otimes \underset{\sim}{\mathbf{h}} + A_5(\underset{\sim}{\mathbf{r}} \otimes \underset{\sim}{\mathbf{h}} + \underset{\sim}{\mathbf{h}} \otimes \underset{\sim}{\mathbf{r}}) \\ & + A_6(\underset{\sim}{\boldsymbol{\pi}} \otimes \underset{\sim}{\boldsymbol{\pi}}_1 + \underset{\sim}{\boldsymbol{\pi}}_1 \otimes \underset{\sim}{\boldsymbol{\pi}}) + A_7 \mathbf{n} \otimes \mathbf{r}^\alpha \otimes \mathbf{n} \otimes \mathbf{r}_\alpha + A_8 h^{\alpha\beta} \mathbf{n} \otimes \mathbf{r}_\alpha \otimes \mathbf{n} \otimes \mathbf{r}_\beta \end{aligned} \tag{7.9}$$

$$\begin{aligned}\underset{\sim}{\mathbf{B}} = {} & B_1 \underset{\sim}{\mathbf{r}} \otimes \underset{\sim}{\boldsymbol{\pi}} + B_2 \underset{\sim}{\boldsymbol{\pi}} \otimes \underset{\sim}{\mathbf{r}} + B_3 \underset{\sim}{\boldsymbol{\pi}}_1 \otimes \underset{\sim}{\mathbf{h}} + B_4 \underset{\sim}{\mathbf{h}} \otimes \underset{\sim}{\boldsymbol{\pi}}_1 + B_5 \underset{\sim}{\boldsymbol{\pi}} \otimes \underset{\sim}{\mathbf{h}} + B_6 \underset{\sim}{\mathbf{h}} \otimes \boldsymbol{\pi} \\ & + B_7 \underset{\sim}{\boldsymbol{\pi}}_1 \otimes \underset{\sim}{\mathbf{r}} + B_8 \underset{\sim}{\mathbf{r}} \otimes \underset{\sim}{\boldsymbol{\pi}}_1 + B_9 \pi^{\alpha\beta} \mathbf{n} \otimes \mathbf{r}_\alpha \otimes \mathbf{n} \otimes \mathbf{r}_\beta + B_{10} \pi_1{}^{\alpha\beta} \mathbf{n} \otimes \mathbf{r}_\alpha \otimes \mathbf{n} \otimes \mathbf{r}_\beta, \end{aligned} \tag{7.10}$$

$$\underset{\sim}{\mathbf{A}}_1 = A_9 \underset{\sim}{\mathbf{r}} + A_{10} \underset{\sim}{\mathbf{h}}, \quad \underset{\sim}{\mathbf{A}}_2 = A_{11} \underset{\sim}{\mathbf{r}} + A_{12} \underset{\sim}{\mathbf{h}}, \quad \underset{\sim}{\mathbf{B}}_1 = B_{11} \underset{\sim}{\boldsymbol{\pi}} + B_{12} \underset{\sim}{\boldsymbol{\pi}}_1, \quad B_2 = B_{13} \underset{\sim}{\boldsymbol{\pi}} + B_{14} \underset{\sim}{\boldsymbol{\pi}}_1 \tag{7.11}$$

where

$$(\underset{\sim}{\mathbf{r}}, \underset{\sim}{\mathbf{h}}) = \mathbf{e}_1 \otimes \mathbf{e}_1 \pm \mathbf{e}_2 \otimes \mathbf{e}_2, \qquad (\underset{\sim}{\boldsymbol{\pi}}, \underset{\sim}{\boldsymbol{\pi}}_1) = \mathbf{e}_1 \otimes \mathbf{e}_2 \mp \mathbf{e}_2 \otimes \mathbf{e}_1 \tag{7.12}$$

and the upper sign corresponds to the first tensor. The tensor $\underset{\sim}{\mathbf{C}}$ may be obtained from (7.9) by replacing A by C.

In deriving (7.9)–(7.11) we have used the fact that during the transformations (7.5), the tensor $\underset{\sim}{\mathbf{e}}$ remains unchanged, but the tensor $\underset{\sim}{\boldsymbol{\kappa}}$ transforms into $-\underset{\sim}{\boldsymbol{\kappa}}$. Thus, it follows that as A does not change, the tensors $\underset{\sim}{\mathbf{A}}$, $\underset{\sim}{\mathbf{C}}$, $\underset{\sim}{\mathbf{A}}_1$, $\underset{\sim}{\mathbf{A}}_2$ are invariant, but the tensors $\underset{\sim}{\mathbf{B}}$, $\underset{\sim}{\mathbf{B}}_1$, $\underset{\sim}{\mathbf{B}}_2$ change sign. It is impossible to simplify the formulae (7.9)–(7.11) for a general directed surface regardless of the physical properties of the surface material. On the other hand, these formulae include by far the most interesting cases. Thus, for example, they hold good for stiffened shells of rather general form. If the physical properties of S_t are transversally isotropic, $\mathbf{n}$ being the axis of isotropy, then for planes and spherical surfaces (under necessary restrictions on $\underset{\sim}{\boldsymbol{\theta}}$), it can be shown that tensors $\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{B}}, \ldots$ must satisfy the following equations

$$(\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{B}}, \underset{\sim}{\mathbf{C}}) = \overset{4}{\underset{1}{\otimes}} \underset{\sim}{\mathbf{Q}} \cdot (\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{B}}, \underset{\sim}{\mathbf{C}}); \quad (\underset{\sim}{\mathbf{A}}_1, \underset{\sim}{\mathbf{A}}_2, \underset{\sim}{\mathbf{B}}_1, \underset{\sim}{\mathbf{B}}_2) = \underset{\sim}{\mathbf{Q}} \cdot (\underset{\sim}{\mathbf{A}}_1, \underset{\sim}{\mathbf{A}}_2, \underset{\sim}{\mathbf{B}}_1, \underset{\sim}{\mathbf{B}}_2) \cdot \underset{\sim}{\mathbf{Q}}^T, \tag{7.13}$$

where

$$\underset{\sim}{Q} = \mathbf{n} \otimes \mathbf{n} + \cos\psi \underset{\sim}{\mathbf{r}} + \sin\psi \underset{\sim}{\boldsymbol{\pi}} \tag{7.14}$$

and

$$\overset{k}{\underset{1}{\otimes}} \underset{\sim}{Q} \cdot \underset{\sim}{S} \equiv S^{\alpha\ldots\beta} \underset{\sim}{Q} \cdot r_\alpha \otimes \cdots \otimes \underset{\sim}{Q} \cdot \mathbf{r}_\beta$$

$\underset{\sim}{S}$ being a tensor of k-th rank and ψ-an arbitrary angle.

Making use of (7.9)–(7.11) and (7.13), we can represent $\underset{\sim}{A}, \underset{\sim}{B}, \ldots$ in the following form

$$\left.\begin{array}{l} \underset{\sim}{A} = A_0 r^{\alpha\beta} \mathbf{n} \otimes \mathbf{r}_\alpha \otimes \mathbf{n} \otimes \mathbf{r}_\beta + A_1 \underset{\sim}{\mathbf{r}} \otimes \underset{\sim}{\mathbf{r}} + A_2 \underset{\sim}{\boldsymbol{\pi}} \otimes \underset{\sim}{\boldsymbol{\pi}} + A_3(\underset{\sim}{\boldsymbol{\pi}}_1 \otimes \underset{\sim}{\boldsymbol{\pi}}_1 + \underset{\sim}{\mathbf{h}} \otimes \underset{\sim}{\mathbf{h}}) \\ \underset{\sim}{B} = B_0 \pi^{\alpha\beta} \mathbf{n} \otimes \mathbf{r}_\alpha \otimes \mathbf{n} \otimes \mathbf{r}_\beta + B_1 \underset{\sim}{\mathbf{r}} \otimes \underset{\sim}{\boldsymbol{\pi}} + B_2 \underset{\sim}{\boldsymbol{\pi}} \otimes \underset{\sim}{\mathbf{r}} + B_3(\underset{\sim}{\boldsymbol{\pi}}_1 \otimes \underset{\sim}{\mathbf{h}} - \underset{\sim}{\mathbf{h}} \otimes \underset{\sim}{\boldsymbol{\pi}}_1) \\ \underset{\sim}{A}_1 = A_4 \underset{\sim}{\mathbf{r}}, \quad \underset{\sim}{A}_2 = A_5 \underset{\sim}{\mathbf{r}}, \quad B_1 = B_4 \underset{\sim}{\boldsymbol{\pi}}, \quad \underset{\sim}{B}_2 = \underset{\sim}{B}_5 \underset{\sim}{\boldsymbol{\pi}}. \end{array}\right\} \tag{7.15}$$

All scalar functions in (7.15) depend on the elastic moduli, the radius of curvature of the carrying surface and the diameter of the "microstructure"—h. It should be noted that there are no surplus functions: there will always exist a material (with transversal isotropy) for which (7.15) will form a minimum complete set. Any further simplification of (7.15) can be made only for a plane, but for this purpose it is necessary to make some additional assumptions. This case is discussed in the next section in which we derive certain formulae which are exact only for a plane.

8. NORMAL ISOTROPIC DIRECTED SURFACE

As mentioned above, further simplification of (7.15) is only possible for planes. In the present section we consider the situation which is, in general, only approximate. At the same time, from the point of view of the three-dimensional theory, it forms the upper limit of validity to a theory that is based on the concepts of forces and couples alone. In effect, we ignore quantities of $0(h^2)$.† To begin with, it is necessary to restrict the class of carrying surfaces that will be considered here. These should be sufficiently smooth, that is in the immediate vicinity of each point of the surface the deviation from the tangential plane is of $0(h^2)$. By immediate vicinity we mean the part of the surface that is included in a circle of radius h. Such a requirement excludes the points of the surface that are situated at a distance of less than h from the ridges and the edges of the surface. The points of the surface that include neighbourhoods with said properties will, if we ignore the effects of $0(h^2)$, be referred to as the points of isotropy. Furthermore, let us assume that the physical properties of a directed surface at the given point are, within the accuracy of terms of $0(h^2)$, invariant with respect to a reversal in the direction of the normal to the tangential plane of the carrying surface. Such points will be referred to as points of normal isotropy, the surfaces formed by these points being called normal isotropic. Thus, for example, a plate of isotropic material is a normal isotropic directed plane, provided the middle plane is chosen as the carrying plane. Insofar as a reversal in the direction of $\mathbf{n} \to -\mathbf{n}$ (herein lies the crux of the definition of a normal that is independent of the surface orientation) does not lead to a sign change in $\underset{\sim}{e}$, the sign change taking place only in $\underset{\sim}{\kappa}$ and the radii of curvature, the tensors $\underset{\sim}{A}$, $\underset{\sim}{C}$, within the terms of $0(h^2)$, are independent of the geometry of the carrying surface, while the tensor $\underset{\sim}{B}$ is a linear function of the radii of curvature. Moreover, if the free energy is to be independent of the choice of orientation on the carrying surface, $\underset{\sim}{B}$ must be a linear function of the mean curvature of the carrying surface. As regards the tensors $\underset{\sim}{A}_1$, $\underset{\sim}{A}_2$, $\underset{\sim}{B}_1$, $\underset{\sim}{B}_2$ it should be borne in mind that a reversal in the direction of the normal results in an interchange in T_1 and T_2.

Thus for normal isotropic deformable surfaces the free energy takes the form

$$\begin{aligned} A = {} & \frac{1}{2}\underset{\sim}{e} : \underset{\sim}{A} : \underset{\sim}{e} + H\underset{\sim}{e} : \underset{\sim}{B} : \underset{\sim}{\kappa} + \frac{1}{2}\underset{\sim}{\kappa} : \underset{\sim}{C} : \underset{\sim}{\kappa} + A_4 \underset{\sim}{\mathbf{r}} : \underset{\sim}{e}(T_1 + T_2 - 2T_0) \\ & + C_4 \underset{\sim}{\boldsymbol{\pi}} : \underset{\sim}{\kappa}(T_1 - T_2) + \frac{1}{2}K[(T_1 - T_0)^2 + (T_2 - T_0)^2], \end{aligned} \tag{8.1}$$

where the tensors $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, defined by (7.15), and the scalar functions A_4, C_4, K depend on the material properties of the surface but not on its geometry.

†The unit of length is taken to be a characteristic linear dimension, such as the minimum radius of curvature.

9. CONSTITUTIVE EQUATIONS FOR HEAT-FLUX VECTOR AND THE EXCHANGED HEAT

We restrict our attention to the case of a normal isotropic directed surface. We prescribe the constitutive equations in the form

$$\mathbf{h}_A = \kappa \text{ grad } T_A, \quad Q = \kappa_1(T_2 - T_1), \quad q_A = \gamma_A[T_{+(-)} - T_A], \tag{9.1}$$

where T_+ corresponds to $A = 1$ and T_- to $A = 2$.

Substituting (9.1) into the inequalities (5.3), we get

$$\kappa \geq 0, \quad \kappa_1 \geq 0, \quad \gamma_A \geq 0. \tag{9.2}$$

Making use of (5.2) and (8.1), we can represent S_A in the following form

$$S_A = -A_4\underset{\sim}{\mathbf{r}}:\underset{\sim}{\mathbf{e}} + (-1)^A C_4\underset{\sim}{\boldsymbol{\pi}}:\underset{\sim}{\boldsymbol{\kappa}} + K(T_A - T_0). \tag{9.3}$$

Equations (5.7), after linearization and by substitution of (9.1), can be written as

$$\nabla \cdot \mathbf{h}_A = 0.5\rho q_0 + \rho(q_A + Q_A) - \rho T_A S_A. \tag{9.4}$$

From these equations and (9.1), (9.3), the equations for temperature fields T_1 and T_2 can easily be obtained.

10. SPECIALIZATION TO CONVENTIONAL SHELL THEORY

To specialize the present approach to the conventional shell theory let us consider the problem of evaluating the elastic moduli of a homogeneous elastic shell. For the isothermal case the free energy takes the simplest form

$$A = \frac{1}{2}\underset{\sim}{\mathbf{e}}:\underset{\sim}{\mathbf{A}}:\underset{\sim}{\mathbf{e}} + H\underset{\sim}{\mathbf{e}}:\underset{\sim}{\mathbf{B}}:\underset{\sim}{\boldsymbol{\kappa}} + \frac{1}{2}\underset{\sim}{\boldsymbol{\kappa}}:\underset{\sim}{\mathbf{C}}:\underset{\sim}{\boldsymbol{\kappa}}. \tag{10.1}$$

If the material of the shell body is a non-polar elastic medium, we can prove that

$$A_2 = C_0 = C_1 = B_0 = B_2 = 0. \tag{10.2}$$

From the conditions of positive definiteness of free energy

$$A \geq 0, \tag{10.3}$$

it follows that

$$A_0 \geq 0, \quad A_1 \geq 0, \quad A_3 \geq 0, \quad C_2 \geq 0, \quad C_3 \geq 0 \tag{10.4}$$

$$A_1C_3 - H^2B_1^2 \geq 0, \quad A_3C_2 - H^2B_3^2 \geq 0. \tag{10.5}$$

Making use of dimensional analysis [10], it is easily shown that

$$\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{A}}(Eh, \nu), \quad \underset{\sim}{\mathbf{B}} = \underset{\sim}{\mathbf{B}}(Eh^3, \nu), \quad \underset{\sim}{\mathbf{C}} = \underset{\sim}{\mathbf{C}}(Eh^3, \nu).$$

As these tensors are linear functions of E, we can write

$$\underset{\sim}{\mathbf{A}}(Eh, \nu) = Eh\underset{\sim}{\mathbf{A}}(\nu), \quad \underset{\sim}{\mathbf{B}}(Eh^3, \nu) = Eh^3\underset{\sim}{\mathbf{B}}(\nu), \quad \underset{\sim}{\mathbf{C}}(Eh^3, \nu) = Eh^3\underset{\sim}{\mathbf{C}}(\nu). \tag{10.6}$$

From these formulae and from the requirement of normal isotropy ($H^2h^2 \ll 1$) it follows that conditions (10.5) will be satisfied if the conditions (10.4) hold good. An exception is the membrane theory of shells, i.e. when $C_2 = C_3 = 0$. In this case, it follows from (10.5), that $B_1 = B_3 = 0$.

Now we have to determine the functions ρ, $\rho\underset{\sim}{\boldsymbol{\theta}}$, A_0, A_1, A_3, C_2, C_3, B_1, B_2. These must be found by experiment, which may be mathematical or physical. It is very simple to determine the mass density ρ of the surface and the inertia tensor $\rho\underline{\theta}$. In order to do this, we consider a part ΔS_t of the carrying surface. The mass Δm and the inertia tensor $\Delta\underline{\mathbf{I}}$ of the corresponding part of the shell $\{z \times \Delta S_t\}$ are defined by

$$\Delta m = \int_{\Delta S_t} \int_{-h/2}^{h/2} \rho_*(1 - 2Hz + Kz^2)\, dz\, d\sigma,$$
$$\Delta\underline{\mathbf{I}} = \int_{\Delta S_t} \int_{-h/2}^{h/2} \rho_*[(\mathbf{r} \cdot \mathbf{r})\mathbf{1} - \mathbf{r} \otimes \mathbf{r}](1 - 2Hz + Kz^2)\, dz\, d\sigma,$$

where ρ_* is the density of the medium and K- the Gaussian curvature of the carrying surface, while $\mathbf{r}$ defines the points in the volume $\{z \times \Delta S_t\}$.

It is obvious that ρ and $\rho\underline{\theta}$ can be found from

$$\rho = \lim_{\Delta S_t \to 0} \frac{\Delta m}{\Delta S_t} = \int_{-h/2}^{h/2} \rho_*(1 - 2Hz + Kz^2)\, dz = \rho h + 0(h^3) \tag{10.7}$$

$$\rho\underline{\theta} = \lim_{\Delta S_t \to 0} \frac{\Delta\underline{\mathbf{I}}}{\Delta S_t} = \int_{-h/2}^{h/2} \rho_* z^2(1 - 2Hz + Kz^2)\, dz(\underline{1} - \mathbf{n} \otimes \mathbf{n})$$
$$= \left[\rho_* \frac{h^3}{12} + 0(h^5)\right](\underline{1} - \mathbf{n} \otimes \mathbf{n}). \tag{10.8}$$

From (10.8) and (10.1) it follows that the so-called sixth equation of motion is no longer a differential one. This means that we have an additional restriction imposed on the constitutive equations. However, the form of these equations does not permit exact satisfaction of this equation of motion except in the case when the structure has the form of a plate or a sphere. However, this is of no consequence to the present theory because of the following.

Let us consider two arbitrary tensor fields $\underline{\epsilon}$ and $\underline{\kappa}$ and evaluate the tensors $\underline{\mathbf{T}}$ and $\underline{\mathbf{M}}$ from the formulae (7.15). Then it is easy to see that the following equations hold good

$$\lim_{h \to 0} [h^{-1} L_i] = 0(1) \neq 0, \qquad i = 1, 2, 3, 4, 5; \tag{10.9}$$

$$\lim_{h \to 0} [h^{-3} L_6] = 0(1) \tag{10.10}$$

where L_i is the ith equation of motion.

We can therefore conclude that the secondary part of the sixth equation, which remains unsatisfied, is associated with second order effects and may be ignored. On the other hand, we have to require the basic operator of the elastic shell theory to be a self-adjoint one. For this it is necessary and sufficient to impose the following restriction

$$\mathbf{n} \cdot \boldsymbol{\phi} = 0 \Rightarrow \phi_3 = 0,$$

as was to be expected, insofar as the material of the shell is a non-polar elastic medium.

Thus, the form of the basic equations and the asymptotic behaviour of their coefficients as $h \to 0$ are known. However, the values of these unknown coefficients are found in an inverse manner by utilizing their unique correspondence to the frequency spectrum of an arbitrary "shell-like" structure, chosen conveniently. We demonstrate this by solving two eigenvalue problems—one of which describes a three-dimensional structure and the other a two-dimensional one.

Problem 1: We are required to determine the eigenfrequencies of an elastic body that occupies the region

$$[-h/2 \le z \le h/2, \quad -a \le x \le a, \quad -b \le y \le b]$$

and is subject to the following boundary conditions

$$\begin{aligned} z &= \pm h/2, \quad \tau_{zz} = \tau_{zx} = \tau_{xy} = 0; \\ x &= \pm a, \quad V = W = 0 \quad \tau_{xx} = 0; \\ y &= \pm b, \quad U = W = 0 \quad \tau_{yy} = 0. \end{aligned}$$

Problem 2: We are required to determine the eigenfrequencies of an elastic directed plane that occupies the region

$$[-a \le x \le a, \quad -b \le y \le b]$$

and is subject to the following boundary conditions

$$\begin{aligned} x &= \pm a, \quad U_2 = U_3 = 0, \quad T_{11} = 0, \quad \phi_1 = 0, \quad M_{12} = 0; \\ y &= \pm b, \quad U_1 = U_3 = 0, \quad T_{22} = 0, \quad \phi_2 = 0, \quad M_{21} = 0. \end{aligned}$$

It is clear from physical reasoning that the problems are almost identical, provided the thickness h is sufficiently small. We shall say that two elastic systems are identical if their corresponding eigenfrequencies are identical.

The solutions of these problems can be found without any difficulty. Thus, making use of (10.4) and (10.6), the eigenfrequencies can be written down in ascending order of magnitude

Theory of Elasticity	Shell Theory	
$\omega = \dfrac{Eh^2}{12(1-\nu^2)}\sigma^4 + 0(h^4)$,	$\omega = \dfrac{C_2 + C_3}{h}\sigma^4 + 0(h^4)$,	(10.11)
$\omega = G\sigma^2$,	$\omega = \dfrac{A_2}{h}\sigma^2$,	(10.12)
$\omega = \dfrac{E}{1-\nu^2}\sigma^2$,	$\omega = \dfrac{A_1 + A_2}{h}\sigma^2$,	(10.13)
$\omega = G\left(\dfrac{\pi^2}{h^2} + \sigma^2\right)$,	$\omega = \dfrac{12C_2}{h^3}\left(\dfrac{A_0}{C_2} + \sigma^2\right)$,	(10.14)
$\omega = G\left(\dfrac{\pi^2}{h^2} + K\sigma^2\right) + 0(h^2)$,	$\omega = G\left(\dfrac{12A_0}{Gh^3} + \dfrac{K_*}{G}\sigma^2\right) + 0(h^2)$,	(10.15)
$\omega = G\left(\dfrac{4S^2\pi^2}{h^2} + \sigma^2\right)$, $S = 1, 2, \ldots,$	—	(10.16)
$\omega = G\left[\dfrac{(2S-1)^2\pi^2}{h^2} + \sigma^2\right]$, $S = 2, 3, \ldots,$	—	(10.17)
$\omega = G\left[\dfrac{(2S-1)^2\pi^2}{h^2} + K_S\sigma^2\right]$, $S = 2, 3, \ldots,$	—	(10.18)

where

$$\sigma^2 = \frac{(2n-1)^2\pi^2}{a^2} + \frac{(2m-1)^2\pi^2}{b^2}; \qquad n, m = 1, 2, \ldots,$$

$K > 1$, $K_S > 1$, $K_* > 12C_2/h^3$ are numbers and $G = E/2(1+\nu)$. The eigenfrequencies of problem 1 are tabulated on the left side and those of problem 2 on the right.

We determine the elastic moduli by demanding coincidence of the frequency dispersion surfaces obtained using the two-dimensional theory with the lowest foils of the corresponding surfaces obtained from the three-dimensional theory of elasticity. Of course, this requirement has a meaning only when $h \to 0$.

From (10.11)–(10.15) it follows that the elastic moduli are determined by the relations

$$\left.\begin{aligned} C_2 + C_3 &= Eh^3/12(1-\nu^2), \qquad C_2 = Gh^3/12, \\ A_1 + A_2 &= Eh/(1-\nu^2), \qquad A_2 = Gh, \\ A_0 &= (\pi^2/12)\cdot Gh. \end{aligned}\right\} \tag{10.19}$$

These relations are well-known in the classical shell theory. The only exception is the modulus of transverse shear A_0, but Reissner's well-known calculations give the value of A_0 in (10.19). The numerical factor $\pi^2/12$ was also obtained by Mindlin using rather similar arguments (see, for example, Naghdi[9]). In order to determine the remaining moduli B_1 and B_3 we may study, for example, the vibration problem of a thin hollow sphere or a hollow cylinder. From the solutions of both these problems it transpires that, within an accuracy of $0(h^2)$, moduli B_1 and B_3 vanish. Moreover, these solutions confirm the formulae (10.19). Generally speaking, the next problem should be to prove that the elastic moduli so obtained do not depend on the boundary conditions. Although the answer is self-evident, a strict proof would be in order. Such a proof can be given by making use of the idea from Weinstein's method of intermediate operators (see, for example, Gould, [7]).

DISCUSSION

Insofar as the model considered here had only five degrees of freedom (at every point), we could describe only a corresponding number of frequency dispersion surfaces for the elastic layer. This means that the theory so developed is valid only in the region of sufficiently low frequencies. Moreover, it is rather interesting to point out that all dispersion surfaces that were discarded (10.16)–(10.18) have the same asymptotic order as those included in the analysis (10.14), (10.15). It follows from this that shell theories that include transverse shear deformations are asymptotically no more accurate than those that ignore such deformations. Therefore, any attempt to construct a shell theory accurate to within $0(h^n)$, where $n \geq 2$, is predestined to fail. Nevertheless, such theories turn out to be both useful and quite accurate. However, estimates of their accuracy cannot be based on a simple manipulation of quantities of $0(h^n)$, but should account for the actual structure and the value of the coefficients (which, of course, depend on the external loads) in the asymptotic expansions. In essence, such methods of estimating the accuracy can no longer be called asymptotic in the usual sense.

In this connection, it should be stressed that in speaking of the second order effects (Section 8) we mean only the inherent estimates of the theory of a directed surface. As a matter of fact, we are unable to point out the accuracy of the given theory in comparison with the three-dimensional theory of elasticity. This may be considered a drawback of the theory. However, from our point of view, the possibility of constructing a theory without resort to the three-dimensional theory of elasticity is, in itself, a major advantage of the present approach. Of course, as in the classical theory of elasticity, we do not touch upon the question of a mathematically exact meaning of such physical concepts as force, couple, displacement, rotation and temperature.

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